Ergodic Theory and Measured Group Theory
Lecture 13

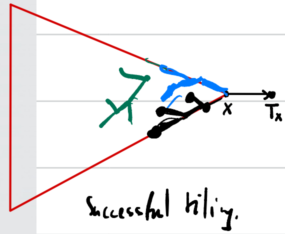
Backward trees theorem (Ʊ.–Zomback 2020+)

A countable-to-one pmp transformation $T : X \to X$ is ergodic if and only if for each $f \in L^1(X, \mu)$ and for a.e. $x \in X$,

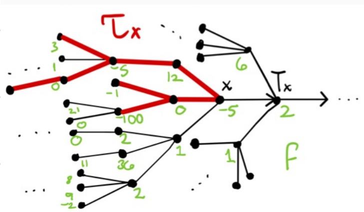
Radon-Nikodym cocycle of E_T wrt μ.

$$\frac{1}{\mathfrak{w}_x(\tau_x)} \sum_{y \in \tau_x} f(y)\mathfrak{w}_x(y) \to \int_X f \, d\mu \quad \text{as} \quad \mathfrak{w}_x(\tau_x) \to \infty,$$

where τ_x ranges over (finite diameter) subtrees of $\mathrm{graph}(T)$ behind x rooted at x.



Successful tiling.



"Proof." Invariance follows from a tree at x plus the edge (x, Tx) still being a tree.

Local-global bridge. $\forall f \in L^1(X,\mu), \forall n,$

$\int_X f \, d\mu = \int (\mathfrak{w}_x\text{-average of } f \text{ over } \triangleright_T^n(x)) \, d\mu(x).$ □

Theorem (Ʊ.–Zomback 2020+)

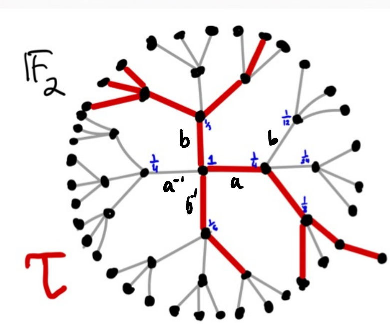
$\langle S \rangle$, where $S := \{s_1^{\pm 1}, \dots, s_r^{\pm 1}\}$

Let $\mathfrak{m}$ be a stationary Markov measure on $\mathbb{F}_r$, $r < \infty$, such that the measure of each $w \in \mathbb{F}_r$ is positive.

A pmp action $\mathbb{F}_r \curvearrowright (X, \mu)$ is ergodic iff for any $f \in L^1(X, \mu)$,

$$\frac{1}{\mathfrak{m}(\tau)} \sum_{w \in \tau} f(w \cdot x)\mathfrak{m}(w) \to \int_X f \, d\mu \text{ as } \mathfrak{m}(\tau) \to \infty \text{ for a.e. } x \in X,$$

where τ ranges over finite rooted subtrees of the Cayley graph of $\mathbb{F}_r$.

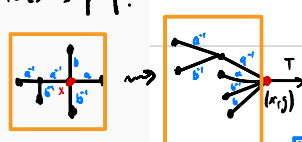


"Proof." Look at the space $X \times S^{\mathbb{N}}$ with measure $\mu \times \mathbb{P}_{\mathfrak{m}}$, where $\mathbb{P}_{\mathfrak{m}}$ is the induced Markov measure.

$T : (X, S^{\mathbb{N}}) \to (X, S^{\mathbb{N}})$ This is pmp.

$(x, y) \mapsto (y_0^{-1} \cdot x, \mathrm{shift}(y)).$

$T^{-1}(x,y) = \{(s \cdot x, sy) : s \in S\}$



Apply the backward theorem. □

The two main kinds of $\mathbb{Z}$-actions. The global goal is to understand pmp actions of ctbl groups up to isomorphism. It turns out that this task is hopeless even for $\mathbb{Z}$. So let's start modestly by defining some properties of $\mathbb{Z}$-actions that can be used to distinguish two such actions.

One such property is ergodicity, but every action of $\mathbb{Z}$ admits an ergodic decomposition, so it makes more sense to focus on ergodic actions and try to distinguish two such actions.

There are two fundamental properties of actions of $\mathbb{Z}$: compactness and weak mixing.

Weak mixing. Before defining, let's state and prove an equivalent condition to ergodicity that is similar to the definition of (strong) mixing, so that then we define weak mixing as something in between.

Recall. A pmp transformation T on (X,μ) is mixing (aka strongly mixing) if $\forall$ Borel sets $A, B \subseteq X$,
$$\mu(T^{-n}(A) \cap B) \underset{n\to\infty}{\longrightarrow} \mu(A)\mu(B).$$

$$|\langle T^n f, g\rangle - \int f \int g| \to 0$$

Equivalently, $\langle T^n f, g\rangle \xrightarrow[n\to\infty]{} \int f\,d\mu \cdot \int g\,d\mu \quad \forall f, g \in L^2(X,\mu)$.

Equivalently, $\langle T^n f, g\rangle \xrightarrow[n\to\infty]{} 0 \quad \forall f, g \in L^2_0(X,\mu)$, where L^2_0 is the space of functions with mean $=0$, i.e. $\int f\,d\mu = 0$.

(last equivalence is by applying $\uparrow$ to functions $f - \int f$ and $g - \int g$.)

Mean ergodic theorem (von Neumann). A pmp T on (X,μ) is ergodic $\iff$ $\forall f, g \in L^2(X,\mu)$,

$$\frac{1}{N+1}\sum_{n=0}^{N}\langle T^n f, g\rangle \xrightarrow[n\to\infty]{} \int f\,d\mu \cdot \int g\,d\mu,$$

i.e.

$$\frac{1}{N+1}\sum_{n=0}^{N}\left(\langle T^n f, g\rangle - \int f\,d\mu \int g\,d\mu\right) \to 0.$$

In particular,

$$\frac{1}{N+1}\sum_{n=0}^{N}\mu(T^{-n}A \cap B) \xrightarrow[n\to\infty]{} \mu(A)\mu(B).$$

Proof. $\Rightarrow$. Recall that we obtained from the pointwise ergodic theorem, the L^2-convergence theorem: $\forall f \in L^2(X,\mu)$,

$$\frac{1}{N+1}\sum_{n=0}^{N} T^n f \xrightarrow[n\to\infty]{L^2} \int f\,d\mu \quad \text{in } L^2.$$

(This theorem can be deduced directly using basic Hilbert space theory.) But inner product with any $g \in L^2$ is a continuous functional $f \mapsto \langle f, g \rangle$, hence

$$\left\langle \frac{1}{N+1} \sum_{n=0}^{N} T^n f, g \right\rangle \xrightarrow[n \to \infty]{} \left\langle \int f \, d\mu, g \right\rangle$$

$$\frac{1}{N} \sum_{n=0}^{N} \langle T^n f, g \rangle \qquad\qquad \int \left(\int f \, d\mu \right) \cdot g \, d\mu = \int f \, d\mu \int g \, d\mu$$

$\Leftarrow$. Taking $B := A$, we get

$$\frac{1}{N+1} \sum_{n=0}^{N} \mu(T^{-n} A \cap A) \to \mu(A)^2.$$

If A is T-invariant, then $T^{-n} A = A$, so $T^{-n} A \cap A = A$, so

$$\frac{1}{N+1} \sum_{n=0}^{N} \mu(T^{-n} A \cap A) = \frac{1}{N+1} \sum_{n=1}^{N} \mu(A) = \mu(A).$$

Thus, $\mu(A) = \mu(A)^2$, so $\mu(A) = 0$ or 1. $\square$

Ergodicity $\Leftrightarrow$ $\frac{1}{N+1} \sum_{n=0}^{N} \left(\mu(T^{-n} A \cap B) - \mu(A) \cdot \mu(B) \right) \to 0 \quad \forall A, B \in X.$

$\Uparrow$

Weak mixing $:\Leftrightarrow$ $\frac{1}{N+1} \sum_{n=0}^{N} \left| \mu(T^{-n} A \cap B) - \mu(A) \cdot \mu(B) \right| \to 0 \quad \forall A, B \in X.$

$\Uparrow$

Mixing $:\Leftrightarrow$ $\left| \mu(T^{-n} A \cap B) - \mu(A) \cdot \mu(B) \right| \to 0 \quad \forall A, B \in X.$

We now argue that weak mixing is "closer" to strong mixing than to just ergodicity.

Prop. T is weakly mixing $\iff$ $\lim_{n\to\infty} |\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| = 0$ off of a set of $n \in \mathbb{N}$ of density 0, i.e. $\exists\, M \subseteq \mathbb{N}$ s.t. $d(M) = 0$ and $\forall \varepsilon > 0$ $|\mu(T^{-n}A \cap B) - \mu(A)\mu(B)| < \varepsilon$ $\forall^\infty n \notin M$.

Def. For $M \subseteq \mathbb{N}$, define its upper density by

$$\bar{d}(M) := \limsup_{N\to\infty} \frac{|M \cap I_N|}{|I_N|}, \text{ where } I_N := \{0,1,\dots,N\}.$$

When the $\lim_{n\to\infty} \frac{|M \cap I_N|}{|I_N|}$ exists, we call it density and denote $d(M)$.

Remark. Since the Borel σ-algebra is countably generated, this M can be chosen independently of the sets A, B.

Lemma. If $M_0, M_1, \ldots \subseteq \mathbb{N}$ of upper density 0, then there is a set $M_\infty \subseteq \mathbb{N}$ s.t. it almost contains each M_n, i.e. $M_n \setminus M_\infty$ is finite.